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Three Conjectures on Shepard Interpolatory Operators*

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After establishing direct and converse approximation theorems for the Shepard interpolatory operators, J. Szabados (*Approx. Theory Appl.* **7**, No. 3, 1991, 63–76) left some open saturation problems ("the most intriguing questions" as he said), which he raised as three conjectures. The present paper proves the second parts of some conjectures, but constructs counterexamples to show that the first parts of three conjectures are not true. The constructive procedure uses some novel ideas and techniques. © 1998 Academic Press

1. INTRODUCTION

Let $C_{[0,1]}$ be the space of all continuous functions on the interval $[0, 1]$ with the norm

$$\|\cdot\| = \max_{0 \leqslant x \leqslant 1} |\cdot|.$$

For $f \in C_{[0,1]}$, the Shepard interpolatory operator $S_{n,\lambda}(f, x)$ is defined as

$$S_{n,\lambda}(f, x) = \frac{\sum_{k=0}^{n} f(k/n)\, |x - k/n|^{-\lambda}}{\sum_{k=0}^{n} |x - k/n|^{-\lambda}}, \qquad \lambda \geqslant 1.$$

This operator has been investigated by some mathematicians (cf. [1–4]). After establishing direct and converse approximation theorems for this

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operator, Szabados [5] left some open saturation problems ("the most intriguing questions" as he said in [5]), which he raised as the following three conjectures:

Conjecture 1. If

$$\|f - S_{n,2}(f)\| = O(n^{-1}),$$

then

$$\int_0^1 t^{-1}\omega(f', t)\, dt < \infty$$

and $f'(0) = f'(1) = 0$ hold.

Conjecture 2. For $1 < \lambda < 2$, then

$$\|f - S_{n,\lambda}(f)\| = O(n^{1-\lambda})$$

implies

$$\int_0^1 t^{-\lambda}\omega(f, t)\, dt < \infty;$$

and

$$\|f - S_{n,\lambda}(f)\| = o(n^{1-\lambda})$$

implies $f(x) \equiv \text{const}$.

Conjecture 3. If

$$\|f - S_{n,1}(f)\| = O(\log^{-1} n),$$

then

$$\int_0^1 t^{-1}\omega(f, t)\, dt < \infty;$$

and if

$$\|f - S_{n,1}(f)\| = o(\log^{-1} n),$$

then $f(x) \equiv \text{const}$.

Unfortunately, the results given in this paper show that the above three conjectures are not completely correct, so that the saturation problems need to be further investigated.

Exactly, in this paper we establish the following.

THEOREM 1. *There is a function f with $f' \in C_{[0,1]}$ such that*

$$\|f - S_{n,2}(f)\| = O(1/n),$$

while

$$\int_0^1 t^{-1}\omega(f', t)\, dt = \infty.$$

THEOREM 2. *For $1 < \lambda < 2$,*

$$\|f - S_{n,\lambda}(f)\| = o(n^{1-\lambda})$$

implies $f(x) \equiv \text{const}$. However, there is a function $f \in C_{[0,1]}$ such that

$$\|f - S_{n,\lambda}(f)\| = O(n^{1-\lambda}),$$

while

$$\int_0^1 t^{-\lambda}\omega(f, t)\, dt = \infty.$$

THEOREM 3. *If*

$$\|f - S_{n,1}(f)\| = o(\log^{-1} n),$$

then $f(x) \equiv \text{const}$. However, there is a function $f \in C_{[0,1]}$ such that

$$\|f - S_{n,1}(f)\| = O(\log^{-1} n),$$

while

$$\int_0^1 t^{-1}\omega(f, t)\, dt = \infty.$$

Remark. We point out that the interest of this paper is not only to answer the conjectures, but also to establish the counterexamples, which themselves show some new techniques and have novelty in constructive analysis. For related matters, interested readers may refer to [6].

2. PRELIMINARIES

To avoid too complicated calculations, we divide some parts of the proofs into several lemmas.

We denote a positive constant by C in the sequel; it may have different values in different situations.

LEMMA 1. *Let $x \in (0, 1)$, and i/n be the nearest node to x, that is,*

$$\min_{k=0, 1, \ldots, n} |x - k/n| = |x - i/n|.$$

Then

$$\sum_{k=0}^{n} |x - k/n|^{-\lambda} \sim |x - i/n|^{-\lambda}, \qquad 1 < \lambda \leqslant 2,$$

$$\sum_{k=0}^{n} |x - k/n|^{-\lambda} \sim (|x - i/n|^{-\lambda} + n), \qquad 0 < \lambda < 1,$$

if $|x - i/n| \sim 1/n$, then

$$\sum_{k=0}^{n} |x - k/n|^{-1} \sim n \log n.$$

Proof. The argument is straightforward. ∎

LEMMA 2. *Let $x \in (0, 1)$. Then there are two subsequences $\{l_k\}$ and $\{n_k\}$ from natural numbers satisfying*

$$\frac{1}{4n_k} \leqslant x - \frac{l_k}{n_k} \leqslant \frac{1}{2n_k}. \tag{1}$$

Proof. We divide the proof into two cases.

Case 1. *$x \in (0, 1)$ is a rational number*. Then x can be written as p/q, where p and q are relative prime, $p \geqslant 2$, $p > q$. Since $(p, q) = 1$, we find two integers u and v such that

$$qu + pv = 1. \tag{2}$$

We also may assume that $u > 0$. Otherwise put $u_1 = u - lp$ and $v_1 = v + lq$, select l to satisfy $u_1 > 0$, and then use u_1, v_1 to replace u, v in (2). Rewrite (2) as

$$\frac{q}{p} u + v = \frac{1}{p},$$

and choose a natural number r with $1/4 \leqslant r/p \leqslant 1/2$. Then we have

$$\frac{1}{4} \leqslant \frac{q}{p} ru + rv = \frac{r}{p} \leqslant \frac{1}{2}. \tag{3}$$

Set $n_k = ur + (k-1)\, pu$, $l_k = k - 1 - v(r + (k-1)\, p)$ (k is a natural number). Then from (2) and (3),

$$\frac{1}{4} \leqslant \frac{q}{p} n_k - l_k = \frac{q}{p} ur + q(k-1)\, u - (k-1) + vr + (k-1)\, pv = \frac{r}{p} \leqslant \frac{1}{2},$$

that is,

$$\frac{1}{4n_k} \leqslant \frac{q}{p} - \frac{l_k}{n_k} \leqslant \frac{1}{2n_k},$$

or (1) holds.

Case 2. *$x \in (0, 1)$ is an irrational number*. Denote the fractional part of x by $\{x\}$, and write $\{x\}$ as a binary number $(0.a_1 a_2 a_3 \cdots)$, where a_i, $i = 1, 2, ...,$ equals 0 or 1. Because x is an irrational number, it must have infinitely many zeros and infinitely many ones. Assume a_{m_k}, $k = 1, 2, ...,$ are infinitely many zeros where each has $a_{m_k+1} = 1$ to follow. Then evidently we have $1/4 < \{2^{m_k - 1} x\} < 1/2$ if we notice $\{2^k x\} = (0.a_{k+1} a_{k+2} a_{k+3} \cdots)$. Thus there are natural numbers q_k satisfying $1/4 < n_k x - q_k < 1/2$ ($n_k = 2^{m_k - 1}$), and equivalently (1) holds.

Altogether, we have completed the proof of Lemma 2. ∎

LEMMA 3. *Let $f \in C_{[0,1]}$, $f \neq \text{const}$. If*

$$\|f - S_{n,\lambda}(f)\| = o(n^{1-\lambda}), \qquad 1 < \lambda \leqslant 2, \tag{4}$$

or

$$\|f - S_{n,1}(f)\| = o(1/\log n), \tag{5}$$

then the maximum and minimum values of $f(x)$ can only be achieved on the endpoints.

Proof. We just prove that the minimum values of f can only be achieved on endpoints. On the contrary, assume $x_0 \in (0, 1)$ is a minimum point of f, and, without loss, assume $f(x_0) = 0$. $f(x)$ must be greater than a given number $\varepsilon_0 > 0$ on an interval $I \subset (0, 1)$ since $f \in C_{[0,1]}$. Denote the length of this interval by $|I|$. By Lemma 2, there are two subsequences $\{l_k\}$ and $\{n_k\}$ from natural numbers satisfying

$$\frac{1}{4n_k} \leqslant x_0 - \frac{l_k}{n_k} \leqslant \frac{1}{2n_k}. \tag{6}$$

Write

$$S_{n_k, \lambda}(f, x_0) - f(x_0) = \frac{\sum_{i=0}^{n_k} f(i/n_k)\, |x_0 - i/n_k|^{-\lambda}}{\sum_{i=0}^{n_k} |x_0 - i/n_k|^{-\lambda}}. \tag{7}$$

For $1 < \lambda \leqslant 2$, by (6), (7), and Lemma 1,

$$\begin{aligned} |f(x_0) - S_{n_k, \lambda}(f, x_0)| &\sim n_k^{-\lambda} \left| \sum_{i=0}^{n_k} f(i/n_k)\, |x_0 - i/n_k|^{-\lambda} \right| \\ &\geqslant C n_k^{-\lambda} \sum_{i/n_k \in I} \varepsilon_0 = C\varepsilon_0\, |I|\, n_k^{1-\lambda}. \end{aligned}$$

This is a contradiction to (4). For $\lambda = 1$, by Lemma 1 again, we get

$$\begin{aligned} |f(x_0) - S_{n_k, 1}(f, x_0)| &= \frac{\sum_{i=0}^{n_k} f(i/n_k)\, |x_0 - i/n_k|^{-1}}{\sum_{i=0}^{n_k} |x_0 - i/n_k|^{-1}} \\ &\sim n_k^{-1} \log^{-1} n_k \left| \sum_{i=0}^{n_k} f(i/n_k)\, |x_0 - i/n_k|^{-1} \right| \\ &\geqslant C n_k^{-1} \log^{-1} n_k \sum_{i/n_k \in I} \varepsilon_0 = C\varepsilon_0\, |I| \log^{-1} n_k, \end{aligned}$$

which contradicts (5). Lemma 3 is proved. ∎

LEMMA 4. *If $f \in C_{[0,1]}$ achieves its minimum value only at $x = 0$, then there is a decreasing sequence $\{x_k\}$, $x_k \to 0$, $k \to \infty$, such that $f(x_k)$ is its minimum value on the interval $[x_k, 1]$.*

Proof. Without loss of generality suppose that $f(0) = 0$. Assume that $f(x)$ achieves its minimum value on $[1/k, 1]$ at $x = x_k$, $k = 1, 2, ...,$ and we will prove such a sequence $\{x_k\}$ is what we require.

Obviously, $\{x_k\}$ is decreasing (it may not be strict). Next, $f(x_k)$ is the minimum value on $[1/k, 1]$, so it is the minimum value on $[x_k, 1]$ as well. If $x_k \not\to 0$ as $k \to \infty$, assume $x_k \to a > 0$ as $k \to \infty$ (we may pass to a subsequence if needed). Now we have $f(a) > 0$. Since $f \in C_{[0,1]}$ and $f(0) = 0$, there is a $\delta > 0$ such that $f(x) < f(a)/2$ for all $x \in [0, \delta)$. Choose sufficiently large k to make $\delta \in (1/k, 1]$. Then we see that, on one hand one has $f(x_k) \leqslant f(x) < f(a)/2$, $x \in [1/k, \delta)$, while $f(x_k) \geqslant f(x_n) \geqslant f(a)$ for $n > k$ on the other hand. Hence $0 < f(a) < f(a)/2$ and this contradiction finishes the proof of this lemma. ▮

LEMMA 5. *If $f' \in C_{[0,1]}$, then for $x \in (0, 1)$,*

$$|f(x) - S_{n,2}(f, x)| = O(n^{-1})\, |f'(x) \log x(1-x)| + O(1) \times \frac{\sum_{k=0}^{n} |f'(\xi_k) - f'(x)|\, |x - k/n|^{-1}}{\sum_{k=0}^{n} (x - k/n)^{-2}},$$

where ξ_k lies between x and k/n.

Proof. The proof is exactly the same as that of [5, Lemma 1]; we have only to notice that

$$f(x) - f(k/n) = f'(x)(x - k/n) + (f'(\xi_k) - f'(x))(x - k/n).$$ ▮

Remark. The inequality

$$|f(x) - S_{n,2}(f, x)| = O(n^{-1})\, |f'(x) \log x(1-x)| + O(n^{-1}) \int_{1/n}^{1} t^{-1} \omega(f', t)\, dt$$

obtained in [5, Lemma 1] is too rough in many cases. Readers can clearly see the benefit in the next section when we construct a counterexample for Conjecture 1 by using the inequality in Lemma 5 instead.

3. PROOFS

Proof of Theorem 2. Suppose $f(x) \not\equiv \text{const}$, by Lemma 3, $f(x)$ can only achieve its maximum and minimum values at the endpoints. Without loss of generality, assume that f only has its minimum value zero at $x = 0$. Applying Lemma 4, we see that there is a decreasing sequence $\{x_k\}$, $x_k \to 0$, $k \to \infty$, such that $f(x_k)$ is its minimum value on the interval $[x_k, 1]$. Therefore we can find an $\varepsilon_0 > 0$ and an interval $I \subset [0, 1]$ such that $f(x) \geqslant \varepsilon_0$ for all $x \in I$. At the same time, for sufficiently large k, $f(x_k) < \varepsilon_0/2$. For such $x_k (< 1/4)$, take n_k satisfying

$$\frac{1}{8n_k} \leqslant x_k \leqslant \frac{1}{4n_k}. \tag{8}$$

We have

$$f(x_k)-S_{n_k,\lambda}(f,x_k) = \frac{f(x_k)\,x_k^{-\lambda}+\sum_{i=1}^{n_k}(f(x_k)-f(i/n_k))\,|x_k-i/n_k|^{-\lambda}}{\sum_{i=0}^{n_k}|x_k-i/n_k|^{-\lambda}},$$

$$f(x_k)-S_{2n_k,\lambda}(f,x_k) = \frac{f(x_k)\,x_k^{-\lambda}+\sum_{i=1}^{2n_k}(f(x_k)-f(i/(2n_k)))\,|x_k-i/(2n_k)|^{-\lambda}}{\sum_{i=0}^{2n_k}|x_k-i/(2n_k)|^{-\lambda}}.$$

From the condition of the theorem

$$\|f-S_{n,\lambda}(f)\|=o(n^{1-\lambda}),$$

together with Lemma 1 and (8), it yields that

$$f(x_k)\,x_k^{-\lambda}+\sum_{i=1}^{n_k}(f(x_k)-f(i/n_k))\left(\frac{i}{n_k}-x_k\right)^{-\lambda}=o(n_k),$$

$$f(x_k)\,x_k^{-\lambda}+\sum_{i=1}^{2n_k}(f(x_k)-f(i/(2n_k)))\left(\frac{i}{2n_k}-x_k\right)^{-\lambda}=o(n_k),$$

and their difference reduces to

$$\sum_{j=1}^{n_k}\left(f\left(\frac{2j-1}{2n_k}\right)-f(x_k)\right)\left(\frac{2j-1}{2n_k}-x_k\right)^{-\lambda}=o(n_k).$$

Now that $f(x_k)<\varepsilon_0/2$, $f(x)>\varepsilon_0$ for all $x\in I$, and $f((2j-1)/(2n_k))>f(x_k)$, with (8), we get

$$\begin{aligned}\frac{\varepsilon_0}{2}\,|I|\,n_k &= O(1)\,\frac{\varepsilon_0}{2}\sum_{(2j-1)/(2n_k)\in I}1\\ &= O(1)\,\frac{\varepsilon_0}{2}\sum_{(2j-1)/(2n_k)\in I}\left(\frac{2j-1}{2n_k}-x_k\right)^{-\lambda}\\ &= O(1)\sum_{(2j-1)/(2n_k)\in I}\left(f\left(\frac{2j-1}{2n_k}\right)-f(x_k)\right)\left(\frac{2j-1}{2n_k}-x_k\right)^{-\lambda}\\ &= o(n_k),\end{aligned}$$

that is,

$$\frac{\varepsilon_0}{2}\,|I|=o(1)$$

as $k \to \infty$, which is impossible. This contradiction arises from the assumption $f(x) \not\equiv \text{const}$; thus we have proved the first part of Theorem 2.

Next we are going to construct a counterexample for the theorem. Select a subsequence of natural numbers $\{n_j\}$ and a sequence of continuous functions $\{f_j(x)\}$ by induction. Let n_1 be a natural number greater than $\max\{4, 1/(2-\lambda)\}$,

$$f_1(x) = \begin{cases} (1/4 - x)^{\lambda - 1 + 1/n_1}, & 0 \leqslant x \leqslant 1/4, \\ 0, & 1/4 < x \leqslant 1. \end{cases}$$

After n_k and $f_k(x)$ are decided, we choose n_{k+1} satisfying

$$n_{k+1} \geqslant 2^{n_k} n_k. \tag{9}$$

Write $N_k = 2 \sum_{j=1}^{k} n_j^{-1}$ and set

$$f_{k+1}(x) = \begin{cases} 0, & x \in [0, N_k], \\ |x - N_k|^{\lambda - 1 + 1/n_{k+1}}, & \\ \quad x \in (N_k, N_k + 1/(2n_{k+1})], & \\ |x - N_k - 1/n_{k+1}|^{\lambda - 1 + 1/n_{k+1}}, & \\ \quad x \in (N_k + 1/(2n_{k+1}), N_k + 1/n_{k+1}], & \\ 0, & x \in (N_k + 1/n_{k+1}, 1]. \end{cases}$$

We observe that $N_{k+1} = N_k + 2/n_{k+1}$, and, with (9),

$$\lim_{k \to \infty} N_k = 2 \sum_{k=1}^{\infty} \frac{1}{n_k} \leqslant 2 \sum_{k=2}^{\infty} 2^{-k} = 1,$$

so that we have well defined continuous functions $\{f_j(x)\}$ on $[0, 1]$ which have the following property: for any $x_0 \in [0, 1]$, if for some j_0, $f_{j_0}(x_0) \neq 0$, then for any $j \neq j_0$, $f_j(x_0) = 0$. By definition, functions $f_k(x)$ also have the other properties

$$t^{\lambda - 1 + 1/n_k}/2 \leqslant |f_k(N_{k-1} + t) - f_k(N_{k-1})| \leqslant \omega(f_k, t) \tag{10}$$

for $0 < t \leqslant 1/(2n_k)$, and

$$\omega(f_k, t) = \max_{0 < h < t} \max_{x \in [N_{k-1}, N_{k-1} + 1/n_k]} |f_k(x + h) - f_k(x)| = O(t^{\lambda - 1 + 1/n_k}) \tag{10'}$$

for $0 < t \leqslant 1$. Define

$$f(x) = \sum_{k=1}^{\infty} n_k^{-1} f_k(x),$$

and write

$$\sum_{j=1,\, j\neq j_0}^{k} n_j^{-1} f_j(x) = F_{k,\, j_0}(x), \qquad \sum_{j=k}^{\infty} n_j^{-1} f_j(x) = R_k(x)$$

for convenience. Obviously $f \in C_{[0,\,1]}$ by

$$\sum_{k=1}^{\infty} n_k^{-1} \|f_k\| \leqslant \sum_{k=1}^{\infty} n_k^{-\lambda} < \infty.$$

Now we estimate $f(x) - S_{n,\lambda}(f, x)$. Let $n_{k-1} < n \leqslant n_k$, $k = 1, 2, ...$ (set $n_0 = 0$), and for some j_0, $x \in I_{j_0} := [N_{j_0 - 1}, N_{j_0 - 1} + 1/n_{j_0}]$ (here $I_1 = [0, 1/4]$). Without loss we assume $j_0 \leqslant k - 1$ (the other case is much easier). Then

$$\begin{aligned} f(x) - S_{n,\lambda}(f, x) &= \frac{\sum_{l=0}^{n} (F_{k-1,\, j_0}(x) - F_{k-1,\, j_0}(l/n))\, |x - l/n|^{-\lambda}}{\sum_{l=0}^{n} |x - l/n|^{-\lambda}} \\ &\quad + \frac{1}{n_{j_0}} \frac{\sum_{l=0}^{n} (f_{j_0}(x) - f_{j_0}(l/n))\, |x - l/n|^{-\lambda}}{\sum_{l=0}^{n} |x - l/n|^{-\lambda}} \\ &\quad + \frac{\sum_{l=0}^{n} (R_k(x) - R_k(l/n)) |x - l/n|^{-\lambda}}{\sum_{l=0}^{n} |x - l/n|^{-\lambda}} \\ &=: \Sigma_1 + \Sigma_2 + \Sigma_3. \end{aligned}$$

It is easy to calculate that

$$\|R_k\| = O\left(\sum_{j=k}^{\infty} n_j^{-\lambda - 1/n_j}\right) = O(n_k^{-\lambda}) = O(n^{-\lambda}),$$

so that

$$|\Sigma_3| \leqslant 2\, \|R_k\| = O(n^{-\lambda}). \tag{11}$$

Dealing with Σ_2 is more complicated. We need the following facts which can be calculated directly. Suppose m_0/n is the nearest node to x. Then for any $j \neq m_0$ (and for $1 \leqslant \lambda \leqslant 2$),

$$|x - j/n|^{-\lambda} \leqslant |j - m_0|^{-\lambda}\, |x - m_0/n|^{-\lambda}. \tag{12}$$

Therefore

$$\left|\sum_{l=0}^{n} (f_{j_0}(x)-f_{j_0}(l/n))\,|x-l/n|^{-\lambda}\right|$$

$$\leqslant \omega(f_{j_0}, n^{-1})\,|x-m_0/n|^{-\lambda}+\sum_{l\neq m_0} \omega\left(f_{j_0}, \frac{|l-m_0|+1}{n}\right)|x-l/n|^{-\lambda}$$

$$\leqslant n^{-\lambda+1-1/n_{j_0}}\left(1+\sum_{l\neq m_0,\,0\leqslant l\leqslant n} (|l-m_0|+1)^{-1+1/n_{j_0}}\right)|x-m_0/n|^{-\lambda}$$

(by (10′) and (12))

$$\leqslant 2n^{-\lambda+1-1/n_{j_0}}\sum_{l=1}^{n} l^{-1+1/n_{j_0}}\,|x-m_0/n|^{-\lambda}$$

$$\leqslant Cn^{-\lambda+1}\,|x-m_0/n|^{-\lambda},$$

together with $\sum_{k=0}^{n}|x-k/n|^{-\lambda}\sim|x-m_0/n|^{-\lambda}$, we thus have

$$|\Sigma_2|=O(n^{-\lambda+1}). \tag{13}$$

Finally, we estimate $|\Sigma_1|$. In this case we notice that $F_{k-1,\,j_0}(x)=0$ since $x\in I_{j_0}$, as well as for any $t\in I_l$, $l\neq j_0$, $1\leqslant l\leqslant k-1$, say, $l<j_0$ (the other case can be treated similarly), $F_{k-1,\,j_0}(t)=n_l^{-1}f_l(t)$. We calculate that

$$J_l:=\left|\sum_{j/n\in I_l}(F_{k-1,\,j_0}(x)-F_{k-1,\,j_0}(j/n))\,|x-j/n|^{-\lambda}\right|$$

$$=\frac{1}{n_l}\sum_{j/n\in I_l}|f_l(j/n)-f_l(N_{l-1}+n_l^{-1})|\,|x-j/n|^{-\lambda}$$

by the structure of $f_l(x)$,

$$f_l(x)\equiv 0 \qquad \text{for} \quad [N_{l-1}+1/n_l, 1].$$

Noticing that $x\in I_{j_0}$ and $j/n\in I_l$, thus $|x-j/n|\geqslant 1/n_l$ and we have

$$J_l\leqslant\frac{1}{n_l}\sum_{j/n\in I_l}|j/n-N_{l-1}-1/n_l|^{\lambda-1+1/n_l}\,|x-j/n|^{-\lambda}$$

$$\leqslant n_l^{\lambda-1}\sum_{j/n\in I_l}|j/n-N_{l-1}-1/n_l|^{\lambda-1+1/n_l}$$

$$\leqslant n_l^{\lambda-1}n^{-\lambda+1-1/n_l}\sum_{j=1}^{[n/n_l]+1} j^{\lambda-1+1/n_l}\leqslant\frac{Cn}{\lambda+n_l^{-1}}\,n_l^{-1-1/n_l}.$$

While by Lemma 1,

$$\sum_{l=0}^{n} |x-l/n|^{-\lambda} \sim |x-m_0/n|^{-\lambda} \geqslant n^{\lambda}.$$

Altogether,

$$|\Sigma_1| = O\left(n^{-\lambda+1} \sum_{j=1}^{\infty} n_j^{-1}\right) = O(n^{-\lambda+1}). \tag{14}$$

Combining (11), (13)–(14), we get

$$f(x) - S_{n,\lambda}(f, x) = O(n^{-\lambda+1}).$$

The remaining part is much simpler:

$$\begin{aligned}
\int_0^1 t^{-\lambda}\omega(f, t)\,dt &\geqslant \sum_{j=1}^{\infty} \int_{n_{j+1}^{-1}}^{2^{-1}n_j^{-1}} t^{-\lambda}\omega(f, t)\,dt \\
&\geqslant \sum_{j=1}^{\infty} \int_{n_{j+1}^{-1}}^{2^{-1}n_j^{-1}} t^{-\lambda}\,|f(N_{j-1}+t) - f(N_{j-1})|\,dt \\
&= \sum_{j=1}^{\infty} \int_{n_{j+1}^{-1}}^{2^{-1}n_j^{-1}} t^{-\lambda}\,|f_j(N_{j-1}+t) - f_j(N_{j-1})|\,dt \\
&\geqslant \sum_{j=1}^{\infty} \frac{1}{n_j} \int_{n_{j+1}^{-1}}^{2^{-1}n_j^{-1}} t^{-1+1/n_j}\,dt \qquad \text{(by (10))} \\
&\geqslant C \sum_{j=1}^{\infty} \left[\left(\frac{1}{2n_j}\right)^{1/n_j} - \left(\frac{1}{n_{j+1}}\right)^{1/n_j}\right] \\
&\geqslant C \sum_{j=1}^{\infty} n_j^{-1/n_j} \qquad \text{(by (9))} \\
&= +\infty.
\end{aligned}$$

Up to this stage, we have completed Theorem 2. ∎

Proof of Theorem 3. The argument of the first part of Theorem 3 is almost the same as that in Theorem 2. The construction is also similar to Theorem 2 (set $\lambda = 1$ in this case). One just needs to notice that in the present case if $|x - j_0/n| \sim n^{-1}$, then (see Lemma 1)

$$\sum_{j=0}^{n} |x - j/n|^{-1} \sim n \log n.$$

The more complicated situation is that, instead of considering the nearest node to x, we now have to consider the second nearest node to x. ∎

Proof of Theorem 1. We construct a counterexample for the theorem. Although the technique is quite similar to that in Theorem 2, some details are different and more complicated. Select a subsequence of natural numbers $\{n_j\}$ and a sequence of continuous functions $\{f_j(x)\}$ by induction. Let $n_1 = 8$,

$$g_1(x) = \begin{cases} 0, & 0 \leqslant x \leqslant 1/8, \\ |x - 1/8|^{1/n_1}, & 1/8 < x \leqslant 5/32, \\ |x - 3/16|^{1/n_1}, & 5/32 < x \leqslant 3/16, \\ -g_1(3/8 - x), & 3/16 < x \leqslant 1/4, \\ 0, & 1/4 < x \leqslant 1. \end{cases}$$

After n_k is decided, we choose n_{k+1} satisfying

$$n_{k+1} \geqslant 2^{n_k} n_k. \tag{15}$$

Write $N_k = 2\sum_{j=1}^{k} n_j^{-1}$ and set

$$g_{k+1}(x) = \begin{cases} 0, \quad x \in [0, N_k], \\ |x - N_k|^{1/n_{k+1}}, \\ \qquad x \in (N_k, N_k + 1/(4n_{k+1})], \\ |x - N_k - 1/(2n_{k+1})|^{1/n_{k+1}}, \\ \qquad x \in (N_k + 1/(4n_{k+1}), N_k + 1/(2n_{k+1})], \\ -g_{k+1}(2N_k + 1/n_{k+1} - x), \\ \qquad x \in (N_k + 1/(2n_{k+1}), N_k + 1/n_{k+1}], \\ 0, \quad x \in (N_k + 1/n_{k+1}, 1]. \end{cases}$$

Set

$$f_k(x) = \int_0^x g_k(t)\, dt.$$

We observe that $N_{k+1} = N_k + 2/n_{k+1}$, and, with (15),

$$\lim_{k \to \infty} N_k = 2 \sum_{k=1}^{\infty} \frac{1}{n_k} < 2 \sum_{k=3}^{\infty} 2^{-k} = 1/2,$$

and $\int_0^x g_k(t)\, dt = \int_{N_{k-1}}^{N_{k-1}+1/n_k} g_k(t)\, dt = 0$ for $x > N_{k-1} + 1/n_k$, so that we have well defined differentiable functions $\{f_j(x)\}$ on $[0, 1]$ which have the following property: for any $x_0 \in (1/8, 1/2)$, if for some j_0, $f_{j_0}(x_0) \neq 0$, then for any $j \neq j_0$, $f_j(x_0)=0$; $f_j(x)=0$ for all $x \in [0, 1/8] \cup [1/2, 1]$ and for all $j=1, 2, \ldots$. By definition, functions $g_k(x)$ also have the other property that

$$\omega(g_k, t) = O(t^{1/n_k})$$

for $0 < t \leqslant 1$. Define

$$f(x) = \sum_{k=1}^{\infty} n_k^{-1} f_k(x),$$

which is the function we required. First,

$$f'(x) = \sum_{k=1}^{\infty} n_k^{-1} g_k(x),$$

together with $\|g_k\| \leqslant 1$ and $\sum_{k=1}^{\infty} n_k^{-1} < \infty$ by (15), we see $f' \in C_{[0,1]}$. Next, similar to the proof of Theorem 2, we can get $\int_0^1 t^{-1}\omega(f', t)\, dt = \infty$. Finally, we will prove that

$$\|f - S_{n,2}(f)\| = O(n^{-1}).$$

Let $n_{k-1} < n \leqslant n_k$, $k=1, 2, \ldots$. Suppose that $x \in (1/8, 1/2)$, say, for some j_0, $x \in I_{j_0} := [N_{j_0-1}, N_{j_0-1} + 1/n_{j_0}]$ (here $I_1 = [1/8, 1/4]$). Without loss we assume $j_0 \leqslant k-1$ (the other case is much simpler). Then

$$\begin{aligned} f(x) - S_{n,2}(f, x) &= \frac{\sum_{l=0}^{n} (F_{k-1, j_0}(x) - F_{k-1, j_0}(l/n))\, |x - l/n|^{-2}}{\sum_{l=0}^{n} |x - l/n|^{-2}} \\ &\quad + \frac{1}{n_{j_0}} \frac{\sum_{l=0}^{n} (f_{j_0}(x) - f_{j_0}(l/n))\, |x - l/n|^{-2}}{\sum_{l=0}^{n} |x - l/n|^{-2}} \\ &\quad + \frac{\sum_{l=0}^{n} (R_k(x) - R_k(l/n))\, |x - l/n|^{-2}}{\sum_{l=0}^{n} |x - l/n|^{-2}} \\ &=: \Sigma_1 + \Sigma_2 + \Sigma_3, \end{aligned}$$

where as in the proof of Theorem 2,

$$\sum_{j=1,\, j \neq j_0}^{k} n_j^{-1} f_j(x) = F_{k, j_0}(x), \qquad \sum_{j=k}^{\infty} n_j^{-1} f_j(x) = R_k(x).$$

Now

$$\|R_k\| = O\left(\sum_{j=k}^{\infty} n_j^{-2-1/n_j} \right) = O(n_k^{-2}) = O(n^{-2}),$$

so that

$$|\Sigma_3| \leqslant 2\,\|R_k\| = O(n^{-2}). \tag{16}$$

To estimate Σ_1, we notice that $F_{k-1,\,j_0}(x)=0$ since $x\in I_{j_0}$, as well as for any $t\in I_l$, $l\neq j_0$, $1\leqslant l\leqslant k-1$, say, $l<j_0$ (the other case can be treated similarly), $F_{k-1,\,j_0}(t)=n_l^{-1}f_l(t)$. Then, due to the clear fact $|f_l(j/n)|\leqslant (j/n-N_{l-1})^{1+1/n_l}$ for $j/n\in I_l$,

$$\begin{aligned}
J_l &:= \left|\sum_{j/n\in I_l} (F_{k-1,\,j_0}(x)-F_{k-1,\,j_0}(j/n))\,|x-j/n|^{-2}\right| \\
&\leqslant \frac{1}{n_l}\sum_{j/n\in I_l} |f_l(j/n)|\,|x-j/n|^{-2} \\
&\leqslant n_l \sum_{j/n\in I_l} |j/n-N_{l-1}|^{1+1/n_l} \\
&\leqslant n_l n^{-1-1/n_l} \sum_{j=1}^{[n/n_l]+1} j^{1+1/n_l}
\end{aligned}$$

similar to the proof of Theorem 2. Therefore,

$$|J_l| \leqslant C n_l n^{-1-1/n_l} [n/n_l]^{2+1/n_l} \leqslant C n_l^{-1-1/n_l} n.$$

With Lemma 1,

$$|\Sigma_1| \leqslant Cn \sum_{j=1,\, j\neq j_0}^{k-1} \frac{1}{n_j}\,|x-m_0/n|^{-2} = O\left(n^{-1}\sum_{j=1}^{\infty} n_j^{-1}\right) = O(n^{-1}), \tag{17}$$

where m_0/n is, as in the proof of Theorem 2, the nearest node to x.

The dealing of Σ_2 is more difficult. By Lemma 5,

$$\begin{aligned}
&\left|\frac{\sum_{l=0}^{n} (f_{j_0}(x)-f_{j_0}(l/n))\,|x-l/n|^{-2}}{\sum_{l=0}^{n} |x-l/n|^{-2}}\right| \\
&\quad = O(n^{-1})\,|g_{j_0}(x)\log x(1-x)| + O(1)\,\frac{\sum_{k=0}^{n} |g_{j_0}(x)-g_{j_0}(\xi_k)|\,|x-k/n|^{-1}}{\sum_{k=0}^{n} (x-k/n)^{-2}} \\
&\quad =: O(n^{-1})\,|g_{j_0}(x)\log x(1-x)| + K,
\end{aligned}$$

where ξ_k lies between x and k/n. Hence it follows that

$$K \leqslant \frac{\omega(g_{j_0}, n^{-1})\, |x-m_0/n|^{-1} + \sum_{l \neq m_0} \omega(g_{j_0}, (|l-m_0|+1)/n)\, |x-l/n|^{-1}}{\sum_{k=0}^{n} (x-k/n)^{-2}}$$

$$\leqslant (x-m_0/n)^2\, n^{-1/n_{j_0}} \left(1 + \sum_{l \neq m_0,\, 0 \leqslant l \leqslant n} (|l-m_0|+1)^{-1+1/n_{j_0}}\right) |x-m_0/n|^{-1}$$

(by Lemma 1 and (12))

$$\leqslant 2\, |x-m_0/n|\, n^{-1/n_{j_0}} \sum_{l=1}^{n} l^{-1+1/n_{j_0}} \leqslant C n_{j_0} n^{-1}.$$

Thus we have

$$|\Sigma_2| = O(n^{-1})\, |g_{j_0}(x) \log x(1-x)| + O(n^{-1}) = O(n^{-1}) \tag{18}$$

since $x \in (1/8, 1/2)$. Combining (16)–(18) yields that

$$f(x) - S_{n,2}(f, x) = O(n^{-1})$$

for $x \in (1/8, 1/2)$.

In case that $x \in (0, 1/8]$ or $x \in [1/2, 1)$, we only notice that $f_k(x) = 0$ for all k in this case, so that we do not need to deal with the summation like the above Σ_2, and clearly we also have

$$f(x) - S_{n,2}(f, x) = O(n^{-1})$$

for $x \in (0, 1/8] \cup [1/2, 1)$. If taking into account the fact that O in the above inequality is independent of x, we have

$$\|f - S_{n,2}(f)\| = O(n^{-1}),$$

so that the proof is finished. ∎

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